# Inverse anticipating chaos synchronization

E. M. Shahverdiev,* S. Sivaprakasam, and K. A. Shore
*School of Informatics, University of Wales, Bangor, Dean Street, Bangor, LL57 1UT, Wales, United Kingdom*
(Received 6 February 2002; published 19 July 2002)

We derive conditions for achieving inverse anticipating synchronization where a driven time-delay chaotic system synchronizes to the inverse future state of the driver. The significance of inverse anticipating chaos in delineating synchronization regimes in time-delay systems is elucidated. The concept is extended to cascaded time-delay systems.

DOI: 10.1103/PhysRevE.66.017204 PACS number(s): 05.45.Xt, 05.45.Vx, 42.55.Px, 42.65.Sf

Chaos synchronization [1] is of fundamental importance in a variety of complex physical, chemical, and biological systems [2]. Synchronization of coupled chaotic systems eliminates some degrees of freedom of the coupled system and so produces a significant reduction of complexity. The occurrence of synchronization between elements of a large system allows significant simplification of computational and theoretical analysis of the system.

Time-delayed systems are ubiquitous in nature, technology, and society because of finite signal transmission times, switching speeds, and memory effects [3]. Therefore the study of chaos synchronization in these systems is of considerable practical significance. Because of their ability to generate high-dimensional chaos, time-delay systems are good candidates for secure communications based on chaos synchronization [4]. Time-delay systems can also be considered as a special case of spatio-temporal systems [5].

In this paper we report a different type of synchronization: inverse anticipating synchronization, where a time-delayed chaotic system $x$ drives another system $y$ in such a way that the driven system anticipates the driver by synchronizing to its inverse future state: $x(t)=-y_{\tau}\equiv -y(t-\tau)$ or equivalently $y(t)=-x(t+\tau)$ with $\tau>0$. *Anticipating synchronization* $x=y_{\tau}$ was discovered by Voss [6] and has recently been the subject of experimental demonstration using external cavity laser diodes [7]. A key result of the present paper is the demonstration of an intimate connection between these two distinct phenomena which arise when different conditions are met. The explicit prescription of the conditions for observing these distinct phenomena offers, first of all, intriguing opportunities for constructive intervention in the dynamical evolution of chaotic systems by switching between the two forms of synchronization.

We investigate inverse anticipating synchronization between two coupled systems for the cases of a single delay time and two characteristic delay times. In the latter case the delay time in the coupling is different from the feedback delay time within the coupled systems themselves.

We define inverse anticipating synchronization as follows. The driver system

$$\frac{dx}{dt}=-\alpha x+f(x_{\tau}) \tag{1}$$

synchronizes with a driven system

$$\frac{dy}{dt}=-\alpha y-f(x) \tag{2}$$

on the inverse anticipating synchronization manifold

$$x=-y_{\tau}. \tag{3}$$

In order to obtain this result we introduce the error signal: $\Delta=x-(-y_{\tau})=x+y_{\tau}$. Then from Eqs. (1) and (2) it follows that $d\Delta/dt=-\alpha\Delta$. In many representative cases, chaos synchronization can be understood from the existence of a global Lyapunov function of the error signals [6,8]. By analyzing the Lyapunov function $L=\frac{1}{2}\Delta^2$ we obtain that for $\alpha>0$ the inverse anticipating synchronization manifold $x=-y_{\tau}$ is globally attracting and asymptotically stable. Throughout this paper, to enhance the accessibility of our presentation, we confine ourselves to the demonstration of principles using specific examples from different areas of physics.

First we consider inverse anticipating synchronization in the following coupled Ikeda systems with a single delay time:

$$\frac{dx}{dt}=-\alpha x-\beta\sin x_{\tau}, \quad \frac{dy}{dt}=-\alpha y+\beta\sin x, \tag{4}$$

where $\alpha>0, \beta>0$. Anticipating synchronization in the coupled Ikeda systems was studied in [6]. The Ikeda model plays an important role in electronics and physiological studies [6]. This model was introduced to describe the dynamics of an optical bistable resonator and is well known for delay-induced chaotic behavior, e.g., [6,9] and references therein. Physically $x$ is the phase lag of the electric field across the resonator and thus may clearly assume both positive and negative values; $\alpha$ is the relaxation coefficient; $\beta$ is the laser intensity injected into the system; $\tau$ is the round-trip time of the light in the resonator. Using the error dynamics approach given above one finds that $x=-y_{\tau}$ is the inverse anticipating chaos synchronization manifold. Numerical simulations fully support the analytical approach. The driver system behaves chaotically for, e.g., $\tau=1$, $\alpha=5$, $\beta=20$. We perform simulations of Eq. (4) by employing an Runge-Kutta-Fehlberg algorithm [10]. Figure 1 shows the time series of the driver $x(t)$ (solid line) and driven system $y(t)$ (dotted line).

*Permanent address: Institute of Physics, 370143 Baku, Azerbaijan.

 

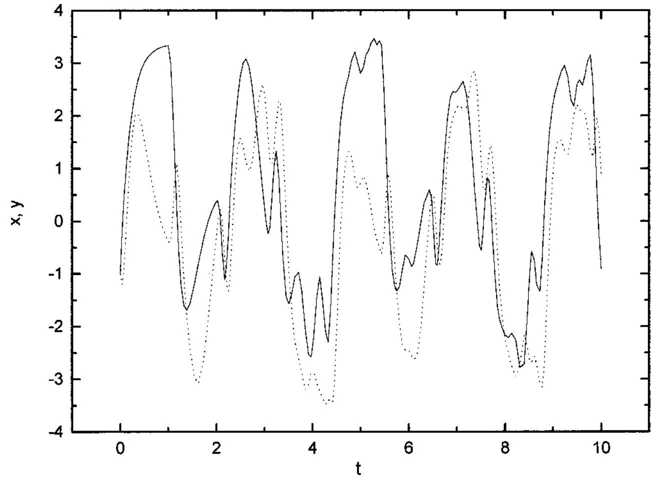


FIG. 1. Numerical simulation of two coupled Ikeda equations: time series of the driver $x(t)$ (solid line) and driven system $y(t)$ (dotted line). After transients (about one time unit), the driven system's trajectory is inverted and shifted one time unit to the left. The parameters of the Ikeda model are $\tau=1$, $\alpha=5$, $\beta=20$. Dimensionless units.

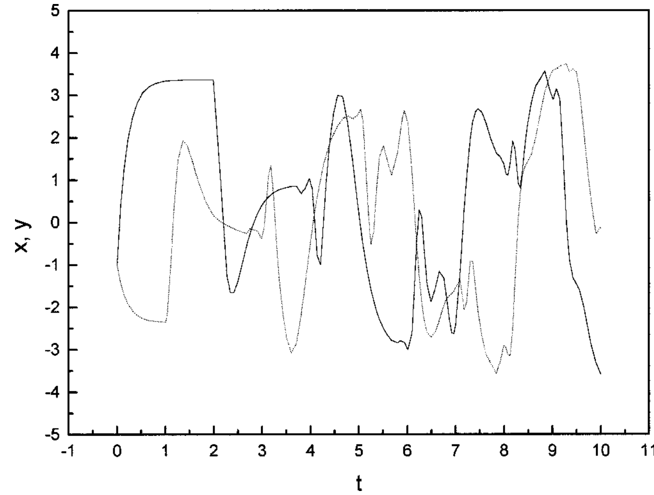


FIG. 2. Numerical simulation of system (6) for $\alpha=5$, $m_1=-18$, $m_2=-3$, $m_3=15$, and for $\tau_1=2$ and $\tau_2=1$: time series of the driver $x(t)$ (solid line) and driven system $y(t)$ (dotted line). After transients, the driven system's trajectory is inverted and shifted $\tau_1-\tau_2=1$ time unit to the left. Dimensionless units.

Inverse anticipating chaos synchronization can also be found in systems with two characteristic delay times: a time delay $\tau_1$ in the coupled systems themselves and a coupling delay time $\tau_2$ between the systems. Consider the following unidirectionally coupled driver $(x)$ and response $(y)$ systems where feedback and coupling delays $\tau_1$ and $\tau_2$ are different.

$$\frac{dx}{dt}=-\alpha x-\beta\sin x_{\tau_1},\quad \frac{dy}{dt}=-\alpha y+\beta\sin x_{\tau_2}. \tag{5}$$

We find that $x=-y_{\tau_1-\tau_2}$ is the inverse anticipating chaos synchronization manifold for Eq. (5) with anticipating time $\tau_1-\tau_2$ $(\tau_2<\tau_1)$, as the error $\Delta=x+y_{\tau_1-\tau_2}$ obeys the following dynamics: $d\Delta/dt=-\alpha\Delta$.

As another example with two characteristic delay times we consider the following delay-coupled Ikeda model [11]:

$$\frac{dx}{dt}=-\alpha x+m_1\sin x_{\tau_1},$$

$$\frac{dy}{dt}=-\alpha y+m_2\sin y_{\tau_1}+m_3\sin x_{\tau_2}, \tag{6}$$

where $m_1$, $m_2$, and $m_3$ are constants. One finds that under the condition

$$m_2=m_1+m_3, \tag{7}$$

Eqs. (6) also admit the inverse anticipating synchronization manifold $x=-y_{\tau_1-\tau_2}$. This follows from the dynamics of the error $\Delta=x+y_{\tau_1-\tau_2}$,

$$\frac{d\Delta}{dt}=-\alpha\Delta+m_2\cos x_{\tau_1}\Delta_{\tau_1}. \tag{8}$$

The sufficient stability condition of the trivial solution $\Delta=0$ of Eq. (8) can be found from the Krasovskii-Lyapunov functional approach for the time-delay systems [3,12,6]: $\alpha>|m_2|$. The condition (7) can be considered as the existence condition for *inverse anticipating chaos synchronization* for the unidirectionally coupled modified Ikeda model. It is noted that the analogous existence condition for *anticipating chaos synchronization* [11] $x=y_{\tau_1-\tau_2}$ is $m_1=m_2+m_3$. We also derive the sufficient stability condition for anticipating synchronization $\alpha>|m_2|$, which coincides with the sufficient stability condition for inverse anticipating synchronization. *Stability conditions found above only hold locally, as Eq. (8) is valid for small $\Delta$; in other words the trivial solution $\Delta=0$ of Eq. (8) is only locally stable.* A stability condition derived from the Krasovskii-Lyapunov approach is a sufficient condition: it assures synchronization for a coupling rate (strength) estimated from the stability condition, but does not forbid the possibility of synchronization with smaller coupling strengths [12].

We would like to emphasize that practical realization of inverse anticipating synchronization in coupled Ikeda systems (6) is rather easily achieved, i.e., the parameters values, e.g., $\alpha=5$, $m_1=-18$, $m_2=-3$, $m_3=15$ satisfy both the existence $m_2=m_1+m_3$ and stability $\alpha>|m_2|$ conditions for the inverse anticipating synchronization manifold. Figure 2 shows numerical simulations of system (6) for the abovementioned values of parameters, and for $\tau_1=2$ and $\tau_2=1$: the time series of the driver $x(t)$ (solid line) and driven system $y(t)$ (dotted line).

We also point out that the existence condition for inverse anticipating synchronization in system (6) coincides with the existence condition for anticipating synchronization in the system obtained from Eq. (6) by replacing $y$ by $-y$ and vice versa. In other words, both system (6) and the system obtained from it after inversion ($y$ by $-y$ or $x$ by $-x$) exhibit both inverse, and "conventional" anticipating synchronization under the indicated existence conditions.

Next we consider complete synchronization

$$x=y \tag{9}$$

between master and slave systems (6). Let us assume $\tau_1=\tau_2$. Then one finds that under the condition $m_1=m_2+m_3$ Eqs. (6) also admit the complete synchronization manifold $x=y$. This conclusion follows from the dynamics of the error $\Delta=x-y$: $d\Delta/dt=-\alpha\Delta+(m_1-m_3-m_2)\sin x_{\tau_1}$ valid for *small* $\Delta$. Thus, the trivial solution of this equation $\Delta=0$ is only locally stable for positive $\alpha$. We observe, therefore, that by changing the feedback, and/or the coupling strengths and feedback delay time one can make transitions between anticipating, inverse anticipating, and complete synchronizations. The significance of this opportunity is underlined by the relative ease for practical implementation of the relevant phenomena using such conveniently operated time-delayed systems as external cavity laser diodes. Studying the possibility of inverse anticipating synchronization in chaotic semiconductor lasers with optical feedback is given added relevance due to the potential for application in secure optical communications [4].

External cavity laser diodes offer an opportunity for accessing inverse anticipating synchronization, which may be manifested in an inversion of the phase of the laser optical field. An appropriate framework for treating the evolution of the electric field of external cavity laser diodes is provided by the widely utilized Lang-Kobayashi equations, see, e.g., [11]. Suppose that a master laser is described by the equations

$$\frac{dE_1}{dt}=\gamma_1(1+\iota\alpha_1)(G_1-1)E_1(t)+k_1E_1(t-\tau_1)\times\exp(-\iota\omega_1\tau_1),$$

$$\frac{dN_1}{dt}=\frac{j_1-N_1-G_1|E_1|^2}{\tau_{n1}}, \tag{10}$$

and is coupled unidirectionally with a slave laser described by the equations

$$\frac{dE_2}{dt}=\gamma_2(1+\iota\alpha_2)(G_2-1)E_2(t)+k_2E_2(t-\tau_1)\times\exp(-\iota\omega_1\tau_1)+k_3E_1(t-\tau_2)\exp(-\iota\omega_1\tau_2),$$

$$\frac{dN_2}{dt}=\frac{j_2-N_2-G_2|E_2|^2}{\tau_{n2}}, \tag{11}$$

where $E_{1,2}$ are the slowly varying complex fields for the master and slave lasers, respectively; $N_{1,2}$ are the normalized carrier densities; $\gamma_{1,2}$ are the cavity losses; $\alpha_{1,2}$ are the linewidth enhancement factors; $G_{1,2}$ are the optical gains; $k_{1,2}$ are the feedback levels; $k_3$ is the coupling rate; $\omega_1$ is the optical frequency without feedback; $\tau_1$ is the round-trip time in the external cavity; $\tau_2$ is the time of flight from the master laser to the slave laser, i.e., the coupling delay time; $j_{1,2}$ are the normalized injection currents; $\tau_{n1,n2}$ are the carrier lifetimes. For identical lasers, inverse anticipating synchronization solutions of Eqs. (10) and (11)

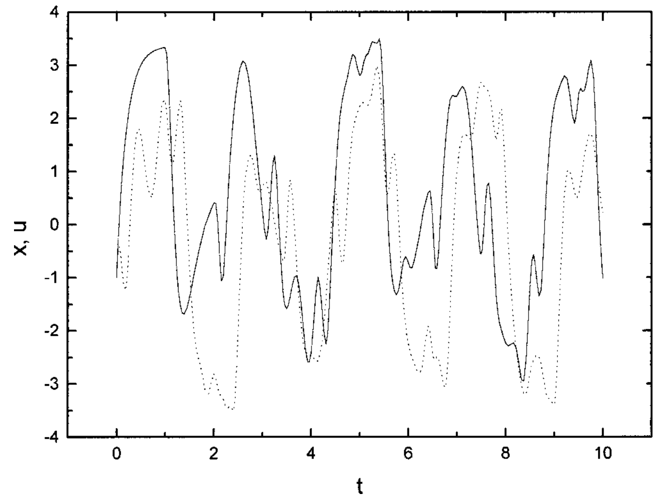


FIG. 3. Numerical simulation of cascaded Ikeda equations: time series of the driver $x(t)$ and driven system $u(t)$. After transients, the driven system's trajectory $u(t)$ is inverted and shifted three time units to the left. The parameters of the Ikeda model are the same as in Fig. 1. Dimensionless units.

$$E_1=-E_{2,\tau_1-\tau_2}, \tag{12}$$

exist if

$$k_2=k_1+k_3. \tag{13}$$

Indeed under condition (13), which is the analog of the sum rule (7), if $E_1$ and $E_2$ are related by

$$E_1(t-\tau_2)\exp(-\iota\omega_1\tau_2)=-E_2(t-\tau_1)\exp(-\iota\omega_1\tau_1), \tag{14}$$

the equations for the optical fields $E_1$ and $-E_2$ are identical and therefore synchronized solutions (12) are possible. From Eqs. (10) and (11) one also can easily derive the existence condition for the complete synchronization manifold $E_1(t)=E_2(t)$: $k_1=k_2+k_3$, which is also the existence condition for the anticipating synchronization manifold $E_1=E_{2,\tau_1-\tau_2}$. We recall that in anticipating synchronization we require $\tau_1>\tau_2$, and in complete synchronization we have $\tau_1=\tau_2$. Comparing "conventional" and inverse anticipating synchronization manifolds written explicitly for the electric field amplitude and phase we notice that a phase shift $\pi$ appears in the case of inverse anticipating chaos synchronization.

Finally we demonstrate that the concept of cascaded synchronization provides increased anticipation times for inverse anticipating chaos synchronization phenomenon. The idea of using cascaded synchronization to increase anticipation times was proposed by Voss [13] in the context of anticipating synchronization between coupled ordinary differential equation systems. We consider the situation when the driven system in Eq. (4) is a chain of three response systems $y$, $z$, and $u$: $dx/dt=-\alpha x-\beta\sin x_\tau$; $dy/dt=-\alpha y+\beta\sin x$; $dz/dt=-\alpha z-\beta\sin y$; $du/dt=-\alpha u-\beta\sin z$. The result is obtained by undertaking an investigation of the error dynamics $\Delta = x-(-z_{2\tau}) = x+z_{2\tau}$: $d\Delta/dt = -\alpha\Delta - \beta(\sin x_\tau + \sin y_{2\tau})$. Given that inverse anticipating synchronization between $x$ and $y$ has already taken place, $x=-y_\tau$ then with $x_\tau = -y_{2\tau}$ we arrive at the error dynamics: $d\Delta/dt = -\alpha\Delta$. From which it is found that the driven system $z$ synchronizes with the driver system $x$ with the anticipation time $2\tau$: $x = -z_{2\tau}$. Thus by adding the new driven system to Eq. (4) the inverse anticipation time is doubled. In the case of three driven systems it is straightforward to find that $x=-u_{3\tau}$ is the inverse anticipating synchronization manifold with the anticipation time $3\tau$. These results are in excellent agreement with numerics. Figure 3 shows numerical simulation of four coupled Ikeda equations: the time series of the driver $x(t)$ and driven system $u(t)$ are presented. After transients, the driven system's trajectory $u(t)$ is inverted and shifted three time units to the left, thus anticipating the driver $x(t)$. It is straightforward to show that cascaded inverse anticipating synchronization also allows one to increase the anticipating times in the case of coupled systems with two characteristic delays thereby providing large anticipating times in a wide class of nonlinear systems—including chaotic external cavity lasers.

To summarize, we have reported a different type of chaos synchronization: inverse anticipating synchronization, where a time-delay chaotic system can drive another system in such a way that the driven system anticipates the driver by synchronizing with its inverse future state. This form of chaos synchronization offers more opportunities for reducing the unpredictability of chaotic dynamics. The utility of the approach is significantly enhanced by the increased inverse anticipation times, which can be obtained in cascaded slave systems.

This work was supported by UK EPSRC under Grant Nos. GR/R22568/01 and GR/N63093/01.

[1] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990); E. Ott, C. Grebogi, and J. A. Yorke, *ibid.* **64**, 1196 (1990).

[2] G. Chen and X. Dong, *From Chaos to Order: Methodologies, Perspectives and Applications* (World Scientific, Singapore, 1998); *Handbook of Chaos Control*, edited by H. G. Schuster (Wiley-VCH, Weinheim, 1999).

[3] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations* (Springer, New York, 1993).

[4] S. Sivaprakasam and K. A. Shore, Opt. Lett. **24**, 466 (1999); H. Fujino and J. Ohtsubo, *ibid.* **25**, 625 (2000); I. Fischer, Y. Liu, and P. Davis, Phys. Rev. A **62**, 011801(R) (2000).

[5] C. Masoller, Chaos **7**, 455 (1997).

[6] H. U. Voss, Phys. Rev. E **61**, 5115 (2000).

[7] S. Sivaprakasam, E. M. Shahverdiev, P. S. Spencer, and K. A. Shore, Phys. Rev. Lett. **87**, 154101 (2001).

[8] R. He and P. G. Vaida, Phys. Rev. A **46**, 7387 (1992); E. M. Shahverdiev, Phys. Rev. E **60**, 3905 (1999).

[9] C. Masoller and D. H. Zanette, Physica A **300**, 359 (2001).

[10] Software for Delay Differential Equations: Time-Delay System URL: Toolbox:http://fde.usaaa.ru

[11] C. Masoller, Phys. Rev. Lett. **86**, 2782 (2001).

[12] K. Pyragas, Phys. Rev. E **58**, 3067 (1998).

[13] H. U. Voss, Phys. Rev. Lett. **87**, 014102 (2001).